Settling a Score by Brimstone   (solution)

1.		2.	3.		4.
8	4	5	4	5	8
	5.	6.			
6	5	3	8	6	1
7.			8.	9.	
6	5	0	2	1	8
10.		11.	12.		13.
1	4	6	1	7	3
	14.				
7	8	5	0	0	6
15.			16.		
8	9	0	9	6	5

ACROSS

1. h_1: $I^2 + J^2$
3. h_2: $B^2 + C^2$
6. h_3: $F^2 + G^2$
7. h_4: $C^2 + F^2 = H^2 + K^2$
8. h_5: $A^2 + B^2$
10. h_6: $G^2 + H^2$
12. h_7: $B^2 + I^2$
14. h_8: $D^2 + J^2$
15. h_9: $F^2 + K^2 = A^2 + J^2$
16. h_{10}: $E^2 + I^2$

DOWN

1. h_{11}: $G^2 + J^2$
2. h_{12}: $B^2 + F^2$
3. h_{13}: $F^2 + H^2$
4. h_{14}: $C^2 + K^2$
5. h_{15}: $G^2 + K^2$
9. h_{16}: $A^2 + H^2$
10. h_{17}: $B^2 + D^2$
11. h_{18}: $H^2 + K^2$
12. h_{19}: $(K-B)^2 + D^2$
13. h_{20}: $F^2 + I^2$

Logical solution path

The table gives the sum of the squares of the first eleven primes nos. E.g. $7^2 + 23^2 = 578$.

	2	3	5	7	11	13	17	19	23	29	31
2	8	13	29	53	125	173	293	365	533	845	965
3	13	18	34	58	130	178	298	370	538	850	970
5	29	34	50	74	146	194	314	386	554	866	986
7	53	58	74	98	170	218	338	410	578	890	1010
11	125	130	146	170	242	290	410	482	650	962	1082
13	173	178	194	218	290	338	458	530	698	1010	1130
17	293	298	314	338	410	458	578	650	818	1130	1250
19	365	370	386	410	482	530	650	722	890	1202	1322
23	533	538	554	578	650	698	818	890	1058	1370	1490
29	845	850	866	890	962	1010	1130	1202	1370	1682	1802
31	965	970	986	1010	1082	1130	1250	1322	1490	1802	1922

The only totals which occur for different pairs of primes are 650 and 890. So 7ac and 11d must be 650 or 890. Also 15ac must be 890 or 650. $890 = 7^2 + 29^2 = 19^2 + 23^2$, $650 = 11^2 + 23^2 = 17^2 + 19^2$
Considering F and K, 7ac and 11d are 650 and 15ac is 890. Also F or C = 17 or 19 H or K = 11 or 23 and A or J = 7 or 29.

By considering other entries in the first grid it is now fairly easy to show that
I=2, D=3, G=5, A=7, H=11, B=13, C=17, F=19, K=23, J=29, E=31

The second grid is most easily filled by making a table of the triangles used. It should be noted that although $a^2 + b^2$ is always the hypotenuse either of $a^2 - b^2$ or $2ab$ can give the shortest side

	s	m	h	
triangle 1	$2IJ = 116$	$I^2 - J^2 = 837$	$I^2 + J^2 = 845$	$116^2 + 837^2 = 845^2$
triangle 2	$C^2 - B^2 = 120$	$2BC = 442$	$B^2 + C^2 = 458$	$120^2 + 442^2 = 458^2$
triangle 3	$2FG = 190$	$F^2 - G^2 = 336$	$F^2 + G^2 = 386$	etc
triangle 4	$F^2 - C^2 = 72$	$2CF = 646$	$C^2 + F^2 = 650$	

etc

1. 8	3	2. 7	3. 4	9	4. 4
3	5. 5	6. 1	4	8	0
7. 2	4	0	8. 2	9. 2	8
10. 1	7	11. 4	12. 1	9	13. 2
1	14. 8	1	6	0	2
15. 6	2	8	16. 5	0	6

ACROSS

1. m_1: $J^2 - I^2$,
3. m_{12}: 2FB
6. $s_{16} + s_{20}$
7. $s_{13} = s_{14}$
8. $s_9 + s_{19}$
10. s_8
12. s_{12}
14. m_{11}
15. $m_{14} - m_{16}$
16. m_{18}

DOWN

1. m_8
2. $s_{13} + s_{14} + s_{15}$
3. m_2
4. s_{18}: $K^2 - H^2$
5. $s_3 + m_{20}$
9. s_{11}
10. s_1: 2IJ
11. m_{13}
12. m_7
13. $m_3 - m_6$